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Effects of field interactions upon particle creation in Robertson–Walker universes

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Abstract. Particle creation due to field interactions in an expanding Robertson–Walker universe is investigated. A model in which pseudoscalar mesons and photons are created as a result of their mutual interaction is considered, and the energy density of created particles is calculated in model universes which undergo a bounce at some maximum curvature. The free-field creation of non-conformally coupled scalar particles and of gravitons is calculated in the same space–times. It is found that if the bounce occurs at a sufficiently early time the interacting particle creation will dominate. This result may be traced to the fact that the model interaction chosen introduces a length scale which is much larger than the Planck length.

1. Introduction

Existing investigations of quantum fields in curved space–time have been mainly confined to massless free fields. In these treatments the only length scale present is the Planck length, so most of the important back-reaction is confined to the Planck regime where the semiclassical approximation of ignoring higher order quantum gravity corrections is invalid. The introduction of a particle mass into the theory provides a further length scale, but this frequently makes little difference as may be seen on dimensional grounds. Massless terms in the stress tensor for the quantum field are typically proportional to R^2, the square of the scalar curvature, whereas mass-dependent terms occur of the form m^2R and m^4. For the latter to dominate the former one demands $m \gtrsim |R|^{1/2}$. In a Robertson–Walker universe this condition is $m \gtrsim t^{-1}$. However, on general grounds one expects important quantum field effects (relative to other matter) only when $|R|^{1/2} \lesssim m$ anyway.

An additional length scale may be introduced into the theory by including the effect of field interactions. In general, one expects that interactions may significantly affect such processes as cosmological particle creation. Birrell and Ford (1979) have recently investigated such effects in particular models. In the present paper we consider additional models for the purpose of comparing interacting field particle creation with free-field particle creation. One might expect that if the interaction introduces a length scale which is large compared with the Planck length, then the interacting particle

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creation will dominate the free-field contribution. Our results support this conjecture. In particular it becomes possible for the universe to undergo a 'bounce' while still many orders of magnitude away from the Planck regime.

2. The π^0–2γ model

A simple model interacting-field theory is that of a massless pseudoscalar field interacting with the electromagnetic field. This is a simplified version of the interaction between neutral pseudoscalar pions and photons, for which a semi-phenomenological theory has long been known. Of course, the theory is probably not renormalisable, and is only an approximation of a more fundamental theory involving quarks, but we take it here at face value, at least to illustrate the possibilities. In any case, questions of renormalisability do not arise in this calculation as only tree diagrams are considered.

In first-order perturbation theory in the absence of gravity, π^0 decays into two γ-rays with a lifetime of about 10^{-16} s (figure 1). When a gravitational field is present, the process shown in figure 2 may occur, in which two photons and a π^0 are simultaneously created from the gravitational field energy. The π^0 subsequently decays, leaving four photons. The process is therefore a source of cosmological photons, and interest attaches to whether this process is more efficient at producing photons than other processes in which conformal symmetry is broken (e.g. by departures from Robertson–Walker symmetry). The assumption that the pion mass is zero

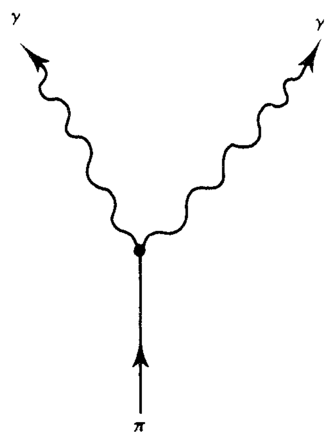

Figure 1. The decay of a π^0 meson into two photons.

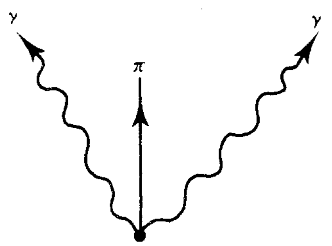

Figure 2. The creation of a π^0 meson and two photons from the vacuum.

should be a good approximation for treating particle creation in regions of space–time where the curvature is larger than about 10^{26} cm^{-2}. In such a situation, the mean energy of the created particles will be large compared to the pion rest mass.

Following Schwinger (1951) we work with an interaction Lagrangian of the form

$$\mathscr{L} = \beta^* F_{\mu\nu} F^{\mu\nu} \phi \tag{2.1}$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor, the asterisk denotes its dual and ϕ is the massless pseudoscalar field. The coupling constant β has dimensions of length, and is known from observations of $\pi^0 \to 2\gamma$ decay to be

$$\beta = 1{\cdot}21 \times 10^{-16}\ \text{cm}. \tag{2.2}$$

We work with a spatially flat Robertson–Walker metric in conformally flat form

$$\mathrm{d}s^2 = a^2(\eta)(\mathrm{d}\eta^2 - \mathrm{d}\boldsymbol{x}^2) \tag{2.3}$$

with a conformally coupled pseudoscalar field so that as far as the free fields are concerned the problem is conformally trivial. Only the coupling breaks the conformal symmetry, and this we can handle using perturbation theory. The first-order S matrix is given by

$$S^{(1)} = \mathrm{i} \int \mathscr{L} \sqrt{-g}\, \mathrm{d}^4x = 4\mathrm{i}\beta \int \sqrt{-g}\, a^{-4}\, (\boldsymbol{E} \cdot \boldsymbol{B})\phi\, \mathrm{d}^4x \tag{2.4}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the Minkowski space electric and magnetic field vectors, respectively. Expanding these in plane waves as usual,

$$\boldsymbol{E} = \mathrm{i} \sum_{\boldsymbol{k},\lambda} \left(\frac{\omega}{2V}\right)^{1/2} \hat{\boldsymbol{e}}_{\boldsymbol{k}}(\lambda)(a_{\boldsymbol{k}\lambda}\, \mathrm{e}^{\mathrm{i}kx} - a^{\dagger}_{\boldsymbol{k}\lambda}\, \mathrm{e}^{-\mathrm{i}kx}) \tag{2.5a}$$

$$\boldsymbol{B} = \mathrm{i} \sum_{\boldsymbol{k},\lambda} \left(\frac{\omega}{2V}\right)^{1/2} \hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{e}}_{\boldsymbol{k}}(\lambda)(a_{\boldsymbol{k}\lambda}\, \mathrm{e}^{\mathrm{i}kx} - a^{\dagger}_{\boldsymbol{k}\lambda}\, \mathrm{e}^{-\mathrm{i}kx}) \tag{2.5b}$$

where V is a normalisation volume, and $\hat{\boldsymbol{e}}_{\boldsymbol{k}}(\lambda)$ is a unit polarisation vector.

Similarly the scalar field may be expanded:

$$\phi = \sum_{k} V^{-1/2}\, a^{-1}\, [b_k \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x} - \mathrm{i}k\eta) + b^{\dagger}_k \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x} + \mathrm{i}k\eta)]. \tag{2.6}$$

We wish to evaluate the S-matrix amplitude for two photons and a pion to be created from the vacuum. This will be

$$\langle \boldsymbol{k}_1, \lambda_1; \boldsymbol{k}_2, \lambda_2; \boldsymbol{k}_3 | S^{(1)} | 0 \rangle = 4\mathrm{i}\beta \int \mathrm{d}^4x \sqrt{-g}\, a^{-4} \langle \boldsymbol{k}_1, \lambda_1; \boldsymbol{k}_2, \lambda_2; \boldsymbol{k}_3 | (\boldsymbol{E} \cdot \boldsymbol{B})\phi | 0 \rangle. \tag{2.7}$$

Substituting (2.5) and (2.6) into (2.7) and performing the integral over the space-like surface orthogonal to the conformal Killing vector $\partial/\partial\eta$, we obtain

$$-2\mathrm{i}\beta \left(\frac{\omega_1\omega_2}{2V\omega_3}\right)^{1/2} \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3,0}\, [\hat{\boldsymbol{e}}_{\boldsymbol{k}_1}(\lambda_1) \cdot (\boldsymbol{k}_2 \wedge \hat{\boldsymbol{e}}_{\boldsymbol{k}_2}(\lambda_2)) + \hat{\boldsymbol{e}}_{\boldsymbol{k}_2}(\lambda_2) \cdot (\boldsymbol{k}_1 \wedge \hat{\boldsymbol{e}}_{\boldsymbol{k}_1}(\lambda_1))] \int_{-\infty}^{\infty} \mathrm{d}\eta\, a^{-1} \exp[\mathrm{i}(\omega_1 + \omega_2 - \omega_3)\eta]. \tag{2.8}$$

The Kronecker delta expresses the conservation of momentum among the created particles.

To compute the number of particle triplets created per unit proper volume we must consider $|\langle S^{(1)}\rangle|^2$. The total number of particles of all momenta and polarisations may then be computed by summing over λ_1 and λ_2, and $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{k}_3$. The polarisation sums may be performed explicitly:

$$\sum_{\lambda\lambda'} [\hat{\boldsymbol{e}}_{\boldsymbol{k}}(\lambda_1)\cdot(\hat{\boldsymbol{k}}_2\wedge\hat{\boldsymbol{e}}_{\boldsymbol{k}_2}(\lambda_2))+\hat{\boldsymbol{e}}_{\boldsymbol{k}_2}\cdot(\hat{\boldsymbol{k}}_1\wedge\hat{\boldsymbol{e}}_{\boldsymbol{k}_1}(\lambda_1))]^2=2(1-\hat{\boldsymbol{k}}_1\cdot\hat{\boldsymbol{k}}_2)^2. \tag{2.9}$$

Thus the number of particles per unit proper volume is

$$n=\frac{4\beta^2}{a^3V^2}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3}\frac{\omega_1\omega_2}{\omega_3}(1-\hat{\boldsymbol{k}}_1\cdot\hat{\boldsymbol{k}}_2)\,\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3,0}|f(\omega_1+\omega_2+\omega_3)|^2 \tag{2.10}$$

where

$$f(\alpha)=\int_{-\infty}^{\infty}\mathrm{d}\eta\, a^{-1}(\eta)\,\mathrm{e}^{-\mathrm{i}\alpha\eta}.$$

Passing to the continuum limit and performing the $\boldsymbol{k}_3$ integration yields

$$n=\frac{\beta^2}{16\pi^6a^3}\int\mathrm{d}^3k_1\,\mathrm{d}^3k_2\frac{\omega_1\omega_2}{\omega_3}(1-\hat{\boldsymbol{k}}_1\cdot\hat{\boldsymbol{k}}_2)|f(\omega_1+\omega_2+\omega_3)|^2 \tag{2.11}$$

where

$$\omega_3=[\omega_1^2+\omega_2^2+2\boldsymbol{k}_1\cdot\boldsymbol{k}_2]^{1/2}.$$

Greater interest attaches to the total density of created energy, ρ. This may be obtained by inserting the factor $a^{-1}(\omega_1+\omega_2+\omega_3)$ in the integrand of (2.11). (In general, $\langle T_{00}\rangle$ contains additional, oscillatory terms which vanish in this case.)

3. Explicit examples

It is of interest to compute ρ for a few explicit space–times. One exactly soluble model is the case

$$a(\eta)=A^2+(\eta/\eta_0)^2 \tag{3.1}$$

where A and η_0 are real constants which represents a universe that contracts to a minimum value A^2 of the scale factor and expands again. In the asymptotic region it behaves like a matter-dominated Friedmann universe. Then

$$f(\alpha)=\pi(\eta_0/A)\exp(-\pi|\alpha\eta_0A|)$$

and

$$\rho=\frac{\eta_0^2\beta^2}{16\pi^4A^4a^4}\int\mathrm{d}^3k_1\,\mathrm{d}^3k_2\left[\frac{\omega_1\omega_2}{\omega_3}(\omega_1+\omega_2+\omega_3)\right.$$
$$\left.\times(1-\hat{\boldsymbol{k}}_1\cdot\hat{\boldsymbol{k}}_2)^2\exp[-2\pi\eta_0A(\omega_1+\omega_2+\omega_3)]\right]. \tag{3.2}$$

On dimensional grounds the integral is of the form: constant$/(\eta_0A)^8$. We used the MACSYMA algebraic manipulation program at MIT to evaluate the integral, which yielded a value $105/4(2\pi)^8$ for the constant. Thus

$$\rho=\frac{105\beta^2}{4(2\pi)^{12}\eta_0^6A^{10}a^4}. \tag{3.3}$$

The answer may be expressed in a more meaningful form by introducing the parameter R_B, the scalar curvature at the 'bounce' where $a(0)=A^2=a_B$. The scalar curvature of the metric equation (3.1) is

$$R=12\eta_0^{-2}(A^2+\eta_0^{-2}\eta^2)^{-3} \tag{3.4}$$

so that

$$R_B=12\eta_0^{-2}A^{-6}. \tag{3.5}$$

Then

$$\rho_{now}=\frac{105\beta^2}{6912(2\pi)^{12}}R_B^3\left(\frac{a_B}{a_{now}}\right)^4. \tag{3.6}$$

Inserting the value (2.2) for β yields

$$\rho_{now}=2\cdot 1\times 10^{-47}R_B^3\,(a_B/a_{now})^4\ \mathrm{g\,cm^{-3}} \tag{3.7}$$

where R_B is in cm^{-2}. For ρ_{now} to be comparable with the present energy density of background photons—about $10^{-34}\ \mathrm{g\,cm^{-3}}$—the bounce must occur at dimensions several orders of magnitude larger than those characterised by the Planck length. For the purpose of obtaining an order of magnitude estimate, we may let $R_B=(3t_B^2)^{-1}$ and $a_B/a_{now}=(t_B/t_{now})^{2/3}$ (although the asymptotic forms $R\sim t^{-2}$ and $a\sim t^{2/3}$ are only strictly valid if $t\gg t_B$). We then find that the bounce is characterised by $t_B\simeq 10^{-26}$ cm $\simeq 10^{-36}$ s, about seven orders of magnitude larger than the Planck length.

Another soluble model is given by the scale factor

$$a(\eta)=[a_B^2+(\eta/\eta_0)^2]^{1/2}. \tag{3.8}$$

If $\eta\gg\eta_0$, this metric approximates that of a radiation-filled Friedmann universe. Here

$$f(\alpha)=2\eta_0K_0(a_B\eta_0\alpha) \tag{3.9}$$

where K_0 is a modified Bessel function. The integrations on d^3k_1 and d^3k_2 may again be performed; the result is of the same form as equation (3.7) aside from the numerical coefficient:

$$\rho=5\times 10^{-43}R_B^3\,(a_B/a)^4. \tag{3.10}$$

(As before, ρ is in $\mathrm{g\,cm^{-3}}$ and R_B in cm^{-2}.) The scalar curvature at $\eta=0$ is in this case given by

$$R_B=6\eta_0^{-2}a_B^{-4}. \tag{3.11}$$

Let us consider a universe in which the metric is given by equation (3.8) until approximately the recombination time. Then the present energy density will be

$$\rho_{now}=\rho_R(a_R/a_{now})^4 \tag{3.12}$$

where $a_R/a_{now}\simeq 10^{-3}$. The energy density at recombination time, ρ_R, is given by equation (3.10) with $a=a_R$. If we again let ρ_{now} be equal to the energy density of the 3K cosmic background, we find that the bounce is characterised by the dimensions of

$$t_B\simeq 10^{-33}\ \mathrm{s}=10^{-23}\ \mathrm{cm}. \tag{3.13}$$

4. Non-conformally invariant free-field particle creation

There are two important features of the π^0–2γ particle creation. One is the non-conformal invariance of the interaction Lagrangian (2.1) which allows particle creation to occur in the massless limit. The other is the presence of a coupling constant with dimensions which introduces an additional length scale into the theory. For comparison let us consider a non-interacting scalar field in Robertson–Walker models with a bounce. Let ϕ satisfy

$$\Box\phi + \xi R\phi = 0. \tag{4.1}$$

If $\xi \neq \frac{1}{6}$, the conformal invariance is broken without the introduction of a length scale. Gravitational wave perturbations in a Robertson–Walker background satisfy (4.1) with $\xi = 0$ (Grishchuk 1974), so this model also describes graviton creation in an expanding universe. It is not possible to treat this problem exactly, so we use the perturbation method of Davies and Unruh (1979) which assumes that $(\xi - \frac{1}{6}) R$ is small. The extent to which this is a good approximation will be discussed below. We shall also present the results of a numerical calculation which does not require this assumption.

Davies and Unruh have evaluated $\langle T_{\mu\nu}\rangle$ at all times for arbitrary Robertson–Walker universes. We specialise their result to late times, where space–time approaches flatness; in this limit the energy density is

$$\rho = -\frac{1}{32\pi^2 a^4}\int_{-\infty}^{\infty} \mathrm{d}\eta_1 \int_{-\infty}^{\infty} \mathrm{d}\eta_2 g'(\eta_1) g'(\eta_2) \ln|\eta_1 - \eta_2| \tag{4.2}$$

where $g = \Lambda R a^2$ and $\Lambda = \xi - \frac{1}{6}$. This is the result for scalar particle creation; the density of gravitons created is obtained by setting $\xi = 0$ and multiplying (4.2) by a factor of 2 to account for the two polarisation degrees of freedom.

If we let the metric be either that given by (3.1) or (3.8), the integral in (4.2) may be evaluated (by use of MACSYMA). The result in both cases is

$$\rho = K\Lambda^2 R_{\mathrm{B}}^2 (a_{\mathrm{B}}/a)^4. \tag{4.3}$$

The numerical coefficient K depends upon the details of the metric; for the scale factor (3.1) it is

$$K(3.1) = \tfrac{1}{128} \approx 7{\cdot}813 \times 10^{-3} \tag{4.4}$$

and for the scale factor (3.8) it is

$$K(3.8) = \tfrac{9}{1024} \approx 8{\cdot}789 \times 10^{-3}. \tag{4.5}$$

Let us now analyse the validity of the perturbation method. It assumes that

$$\lim_{\eta\to\infty} |I| \ll 1 \tag{4.6}$$

where

$$I \equiv k^{-1}\int_{-\infty}^{\eta} g(\eta_1)\,\mathrm{e}^{-ik\eta_1} \sin k(\eta - \eta_1)\,\mathrm{d}\eta. \tag{4.7}$$

Let us consider the particular metric given by equation (3.1). Here

$$I = \frac{1}{2ik}(e^{ik\eta} I_1 - \mathrm{e}^{-ik\eta} I_2) \tag{4.8}$$

where, as $\eta \to \infty$,

$$I_1 = 12|\Lambda|\eta_0^{-2} \int_{-\infty}^{\infty} \frac{e^{-2ik\eta_1}\, d\eta_1}{A^2 + \eta^2\eta_0^{-2}} = 12|\Lambda|\pi\eta_0^{-1} A^{-1} e^{-2\eta_0 Ak} \tag{4.9}$$

and

$$I_2 = I_1|_{k=0} = 12|\Lambda|\pi\eta_0^{-1} A^{-1}. \tag{4.10}$$

Thus

$$\lim_{\eta\to\infty} |I| \leq \frac{1}{2k}(|I_1| + |I_2|) \leq 12\pi|\Lambda|(k\eta_0 A)^{-1}. \tag{4.11}$$

If $\xi = 0$, then the criterion (4.6) is satisfied for values of k such that

$$k \geq 2\pi(\eta_0 A)^{-1}. \tag{4.12}$$

This is of the order of the typical momenta associated with the particles created in this model. Hence the perturbation treatment is at the edge of its limits of validity in this application.

An alternative means of obtaining the free particle energy density is to use the momentum space formulation of Birrell (1979). In this approach the energy density is given in terms of an integral involving a 'T matrix'. This T matrix is obtained as a solution of a Lippmann–Schwinger type equation with potential equal to the Fourier transform of the quantity g in (4.1). One method of approximately solving this Lippmann–Schwinger equation is to use the first Born approximation. This can be shown to yield the approximate energy density (4.1). Another, much more accurate, technique is to solve the equation numerically, as discussed by Birrell (1979). This has been done for the case $\xi = 0$ and the scale factors (3.1) and (3.8), giving results which we believe to be accurate to within about 10%. The numerical results give the proportionality coefficient in (4.2) as

$$K(3.1) = 5{\cdot}4 \times 10^{-2} \tag{4.13}$$

and

$$K(3.8) = 3{\cdot}4 \times 10^{-3} \tag{4.14}$$

for the scale factors (3.1) and (3.8), respectively. Comparison of these coefficients with those in (4.4) and (4.5) indicates that for the purpose of comparison with the density due to $\pi^0 - 2\gamma$ production the Born approximation is sufficiently accurate.

Comparison of (3.6) with (4.3) shows that the free-field particle production is below that in the π^0–2γ model unless $R_B \leq 10^{54}$ cm^{-2}, or the associated characteristic length l is greater than 10^{-27} cm. Let us consider the case where the bounce is characterised by Planck dimensions: $R_B \approx 10^{66}$ cm^{-2}. Then (4.3) may be written as

$$\rho \approx (10^{90}\ \mathrm{g\,cm^{-3}})\,(a_B/a)^4. \tag{4.15}$$

Thus the energy density of particles at any time is equal to that obtained by assuming a density of about $10^{-4}\rho_p$ (ρ_p = Planck density = 10^{94} g cm^{-3}) at the Planck time and then red-shifting forward to the given time. The energy density (4.15) represents, at the present time, a density comparable to that of the 3K cosmic background. This result is to be compared with that of § 3 where it was found that the same present energy density could be created in a universe which bounces at dimensions of the order of 10^{10} Planck lengths. The much larger result in the interacting field model may be traced to the presence of an additional length scale in the theory.

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